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Multivalued Maps and the Existence of Best Approximants

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Some existence of the best approximants in normed linear space is presented. A fixed point theorem for uppersemicontinuous multivalued maps with nonconvex values is used for this purpose. Previous results due to different authors are covered and extended through our method. © 1991 Academic Press, Inc.

INTRODUCTION

In [3] Ky Fan proved that for any continuous function f from a compact convex subset M of a normed linear space X into X there exists $x \in M$ such that

$$\|x - f(x)\| = d(f(x), M).$$

Ky Fan's theorem has been of great importance in Nonlinear Analysis, Approximation Theory, Game Theory, and Minimax Theorems.

There have appeared several extensions of this theorem (see, e.g., [6–8]).

In [6, 8] Ky Fan's theorem has been extended for a continuous onto and almost-affine (see definition below) map instead of the identity map.

The main tools used in those papers are fixed point theorems for suitable multivalued maps: the hypothesis of almost-affinity ensures that those multivalued maps have convex values.

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In the present paper we obtain results for a wider class of maps including the almost-affine ones.

We obtain a multivalued map which is not convex-valued but which is the composition of uppersemicontinuous acyclic valued maps. To achieve our aim we shall use a fixed point theorem of Powers [5] for a class of non-acyclic (hence not convex) multivalued maps.

It seems that this is the first time that a fixed point theorem for multivalued maps without convex values is used for Ky Fan's type theorem.

Finally we give a condition very similar to Prolla [6], one which gives a multivalued map with star-shaped values.

Notations, Definitions, and Preliminary Results

Let X be a Banach space; denote by $\|\cdot\|$ its norm.

Let A be a subset of X and let $x \in X$; denote by

$$d(x, A) = \inf\{\|x - y\|, y \in A\}.$$

A subset $A \subset X$ is called *approximatively compact* if for every $x \in X$ and every sequence $\{x_n\} \subset A$ with $\lim_{n \to \infty} \|x - x_n\| = d(x, A)$, there exists a subsequence $\{x_{n_k}\}$ converging to an element of A [9]. Clearly a compact set is approximatively compact. However, a convex and closed subset of a uniformly convex Banach space is approximatively compact but not compact.

Let M be a convex subset of X. A map $g: M \to X$ is said to be *almost affine* if for all $t_1, t_2 \in M$ and for all $y \in X$, we have

$$\|g(\lambda t_1 + (1 - \lambda) t_2) - y\| \leqslant \lambda \|g(t_1) - y\| + (1 - \lambda) \|g(t_2) - y\|,$$

where $\lambda \in (0, 1)$ (see [6]).

$g: M \to X$ is said to be *affine* if for all $t_1, t_2 \in M$

$$g(\lambda t_1 + (1 - \lambda) t_2) = \lambda g(t_1) + (1 - \lambda) g(t_2), \qquad \lambda \in (0, 1).$$

Clearly an affine map is an almost-affine map.

Let A, B be subsets of X. A multivalued map $F: A \multimap B$ with nonempty and compact values is said to be *uppersemicontinuous* (*u.s.c.*) in A if $\{x \in A : F(x) \subset V\}$ is open in A whenever $V \subset B$ is open.

A multivalued map $F: A \multimap B$ is said to be *compact* if $F(A)$ is contained in a compact subset of B.

Let $g: A \to B$ be a single-valued *proper map* (i.e., $g^{-1}(K)$ is compact whenever K is compact).

Assume that g is onto; then we can define the multivalued map

$g^{-1}: B \multimap A$. Since g is proper, than g is closed (see [2, p. 55]). So that g^{-1} is uppersemicontinuous (see [2, p. 301]).

A subset A of X is said to be *star-shaped* if there exists $x_1 \in X$ such that, for every $x_2 \in X$, $\lambda x_1 + (1-\lambda) x_2 \in X$, $\lambda \in [0, 1]$.

A topological space T is said to be *acyclic* if $\tilde{H}^n(T) = \{0\}$ for every $n \geqslant 0$, where $\tilde{H}^n$ stands for the reduced Čech cohomology with coefficients in $\mathbb{Z}$.

Clearly, a convex or a star-shaped subset of X is acyclic.

A multivalued map $F: A \multimap B$ is said to be *acyclic* (or *acyclic-valued*) if $F(x)$ is a nonempty compact and acyclic subset of B for every $x \in A$.

Let Y be a metric space. A multivalued map $F: Y \multimap Y$ is said to be *admissible* (see [7]) if there are maps $F_i: Y_i \multimap Y_{i+1}$, $i = 0, 1, ..., n$, $Y_0 = Y_{n+1} = Y$ such that:

(1) $F = F_n \circ F_{n-1} \circ \cdots \circ F_0$;

(2) F_i is acyclic and u.s.c. for each $i = 0, 1, ..., n$;

(3) Y_i are metric spaces for each $i = 0, 1, ..., n$.

Notice that the composition of u.s.c. maps is an u.s.c. map. However, if f_1 is a single-valued map and F_2 is an acyclic map, then $f_1 \circ F_2$ is not in general an acyclic map. Hence the composition of acyclic maps is not an acyclic map.

We have the following theorem [5].

THEOREM A. *Let M be a convex subset of a Banach space X and $F: M \multimap M$ an admissible compact map. Then F has a fixed point.*

RESULTS

THEOREM. *Let X be a Banach space and let M be an approximatively compact and convex subset of X. Let $f: M \to X$ be a continuous map such that $f(M)$ is relatively compact. Let $g: M \to M$ be a continuous onto and proper map such that $g^{-1}(z)$ is an acyclic subset of M for every $z \in M$.*

Then there exists $x \in M$ such that

$$\|g(x) - f(x)\| = d(f(x), M). \tag{KF}$$

Proof. Consider the metric projection $Q: X \multimap M$ defined by

$$Q(x) = \{y \in M: \|y - x\| = d(x, M)\}.$$

We have that $Q(x) \neq \varnothing$ for every $x \in X$ and Q sends compact sets into compact sets (see [8]).

Moreover Q is u.s.c. (see [7]).

It is easy to see that, since M is convex, $Q(x)$ is convex for every $x \in X$.

From our hypotheses we obtain that $g^{-1}: M \multimap M$ is an u.s.c. acyclic-valued map. Then the map $G = g^{-1} \circ Q \circ f: M \multimap M$ is an admissible map.

To prove our result it is enough to show that G has a fixed point.

In fact, $f(M)$ is relatively compact so that $G(M) = g^{-1}(Q(f(M)))$ is also relatively compact because the image of a compact set under an u.s.c. map (with compact values) is compact.

By Theorem A it follows that G has a fixed point and the theorem is proved.

Remark 1. An almost affine map $g: M \to M$ is an example of a map such that $g^{-1}(z)$ is an acyclic set for each $z \in M$.

In fact $g^{-1}(z)$ is a convex subset of M. To see this, let $x_1, x_2 \in g^{-1}(z)$ (i.e., $g(x_1) = g(x_2) = z$). We want to show that $g(x) = z$ with $x = \lambda x_1 + (1-\lambda)x_2$, $\lambda \in (0, 1)$.

From the definition we have that, for all $y \in M$,

$$\|g(x) - y\| \leqslant \lambda \, \|g(x_1) - y\| + (1-\lambda)\| \, g(x_2) - y\|.$$

Choose $y = z$. We have $\|g(x) - z\| \leqslant 0$, hence $g(x) = z$.

Remark 2. From Theorem 1 and Remark 1 we obtain as a particular case the result of Sehgal and Singh [8] (see also [7]).

As a consequence of Theorem 1 we have the following Corollary.

COROLLARY 1. *Let X be a Banach space and let M be a compact and convex subset of X. Let $f: M \to X$ be a continuous map and let $g: M \to M$ be a continuous onto map such that $g^{-1}(z)$ is an acyclic subset of M for every $z \in M$. Then there exists $x \in M$ such that* (KF) *holds.*

Remark 3. If $g = I$ we obtain the well known theorem of Ky Fan [3].

Remark 4. There exist several examples of continuous maps such that g^{-1} is an acyclic-valued map (see, e.g., [1, 4]).

Remark 5. Corollary 1 is an extension of the result of Prolla [6]. Notice that in the map paper of Prolla [6] a different multivalued map is used. This multivalued map is convex-valued, since g is almost affine.

Now we introduce a class of maps $g: M \to M$ such that $g^{-1}(z)$ is a star-shaped set, for any $z \in M$. This definition represents an extension of the notion of almost-affine maps (see [6]).

Let M be a convex subset of X. We say that $g: M \to M$ is *star-affine* if for every $y \in X$ there exists $t_1 \in M$, t_1 depending on y, such that for every $t_2 \in M$ and for every $\lambda \in (0, 1)$ we have

$$\|g(\lambda t_1 + (1-\lambda)\, t_2) - y\| \leqslant \lambda\, \|g(t_1) - y\| + (1-\lambda)\, \|g(t_2) - y\| \qquad (\star)$$

and

$$\|g(t_1) - y\| = d(y, M). \qquad (\star\star)$$

Assume that M is an approximatively compact and convex subset of X, then $g: M \to M$ onto and almost affine is star-affine. In fact, there exists $u \in M$ such that $\|u - y\| = d(y, M)$: since g is onto, then ($\star\star$) holds.

Condition ($\star$) is trivially satisfied.

Now we want to show that $g^{-1}(z)$ is a star-shaped set for any $z \in M$. Let $z \in M$, M as above; then there exists $t_1 \in M$ such that, for every $t_2 \in M$ and $\lambda \in (0, 1)$, we have

$$\|g(\lambda t_1 + (1-\lambda)\, t_2) - z\| \leqslant \lambda\, \|g(t_1) - z\| + (1-\lambda) \|\, g(t_2) - z\|$$

and

$$\|g(t_1) - z\| = d(z, M) = 0.$$

This implies that

$$\|g(\lambda t_1 + (1-\lambda)\, t_2) - z\| = 0.$$

So that

$$\lambda t_1 + (1-\lambda)\, t_2 \in g^{-1}(z).$$

Hence g^{-1} is star-shaped.

As a consequence of this fact and our Theorem, we obtain the following result.

COROLLARY 2. *Let M be a convex and approximatively compact subset of a Banach space X. Assume that g: $M \to M$ is a continuous onto proper and star-affine map. Let f: $M \to M$ be a continuous map; then there exists $x \in M$ such that* (KF) *holds.*

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REFERENCES

1. F. E. BROWDER AND C. P. GUPTA, Topological degree and nonlinear mappings of analytic type in Banach spaces, *J. Math. Anal. Appl.* **26** (1969), 390–402.
2. K. DEIMLING, "Nonlinear Functional Analysis," Springer-Verlag, New York/Berlin, 1985.
3. KY FAN, Extension of two fixed point theorems of F. E. Browder, *Math. Z.* **112** (1969), 234–240.
4. J. M. LASRY AND R. ROBERT, Analyse non-linéaire multivoque, *in* "Cahíers de Mathématiques de la Décision," Vol. 761, Univ. Paris IX, 1976.
5. M. J. POWERS, Leftschetz fixed points theorems for a new class of multivalued maps, *Pacific J. Math.* **42**, No. 1 (1972), 211–220.
6. J. B. PROLLA, Fixed point theorems for set-valued mappings and existence of best approximants, *Numer. Funct. Anal. Optim.* **5**, No. 4 (1982–1983), 449–455.
7. S. REICH, Approximate selections, best approximation, fixed point and invariant sets, *J. Math. Anal. Appl.* **62** (1978), 104–113.
8. V. M. SEHGAL AND S. P. SINGH, A theorem on best approximations, *Numer. Funct. Anal. Optim.* **10**, No. 182 (1989), 181–184.
9. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspace," p. 389, Springer-Verlag, New York/Berlin, 1970.